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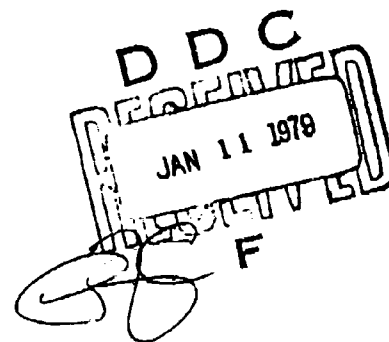
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SEARCH FOR A MOVING TARGET:  
UPPER BOUND ON DETECTION PROBABILITY

by

Alan R. Washburn

October 1978

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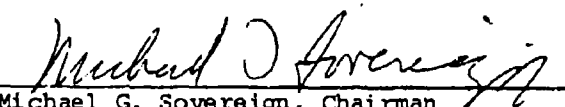
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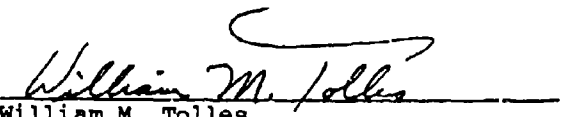
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SEARCH FOR A MOVING TARGET:  
UPPER BOUND ON DETECTION PROBABILITY

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ABSTRACT

Existing algorithms for computing the optimal distribution of effort in search for a moving target operate by producing a sequence of progressively better distributions. This report shows how to compute an upper bound on the detection probability for each of those effort distributions in the case where the detection function is concave.

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## 1. INTRODUCTION

The object is to detect a randomly moving target at one of the discrete times  $0, 1, \dots, \tau$ . The searcher determines a non-negative effort distribution  $\psi(x, t)$  such that the total effort applied at time  $t$  does not exceed  $m(t)$ . Our purpose here is to establish an upper bound on the detection probability for every effort distribution. This is intended to supplement existing iterative procedures that develop sequences of effort distributions that improve monotonically.

## 2. THE GENERAL CASE

Let  $w(x, t)$  be an "effectiveness coefficient" for search effort applied at position  $x$  at time  $t$ . If  $X_t$  is the position of the target at time  $t$ , then the probability of detection depends, we assume, only on the total effective search effort

$$Z = \sum_{t=0}^{\tau} w(X_t, t) \psi(X_t, t) .$$

Specifically the probability of detection is  $P(\psi) = E(b(Z))$ , where the expectation operator is needed because  $X_t$  is a stochastic process. We assume that  $b(z') - b(z) \leq s(z)(z' - z)$  for some function  $s(z)$  and for all  $z'$ ; if the detection

function  $b$  is concave and differentiable, then  $s(z)$  is just  $\frac{d}{dz} b(z)$ . In most applications,  $b(z) = 1 - \exp(-z)$  and  $s(z) = \exp(-z)$ .

Let  $\psi$  and  $\psi'$  be two effort distributions. Then

$$(1) \quad P(\psi') - P(\psi) = E(b(Z') - b(Z)) \leq E(s(Z) (Z' - Z))$$

We also have

$$(2) \quad E(s(Z) (Z' - Z)) = \sum_{t=0}^T E(s(Z) w(X_t, t) [\psi'(X_t, t) - \psi(X_t, t)])$$

We now consider two cases, depending on whether  $X_t$  is discrete or continuous. If  $X_t$  is discrete, we let  $p_t(x)$  be the probability mass function of  $X_t$ , and require

$$\sum_x \psi'(x, t) = \sum_x \psi(x, t) = m(t).$$

We define

$$(3) \quad D_\tau(\psi, x, t) = w(x, t) p_t(x) E(s(Z) | X_t = x)$$

Then (2) can be written

$$(4) \quad E(s(Z) (Z' - Z)) = \sum_{t=0}^T \sum_x D_\tau(\psi, x, t) [\psi'(x, t) - \psi(x, t)]$$

If  $X_t$  is continuous, let  $p_t(x)$  be the probability density function of  $X_t$ , and require  $\int \psi'(x,t)dx = \int \psi(x,t)dx = m(t)$ . Then (2) can be written

$$(4') \quad E(s(Z)(Z'-Z)) = \sum_{t=0}^{\tau} \int D_{\tau}(\psi, x, t) [\psi'(x, t) - \psi(x, t)] dx.$$

In either case, suppose that  $D_{\tau}(\psi, x, t) \leq \bar{\lambda}(t)$  for all  $x$ , and  $D_{\tau}(\psi, x, t) \geq \underline{\lambda}(t)$  when  $\psi(x, t) > 0$ . In the discrete case, from (4),

$$\begin{aligned} (5) \quad E(s(Z)(Z'-Z)) &\leq \sum_{t=0}^{\tau} \left[ \sum_x \bar{\lambda}(t) \psi'(x, t) - \sum_x \underline{\lambda}(t) \psi(x, t) \right] \\ &= \sum_{t=0}^{\tau} (\bar{\lambda}(t) - \underline{\lambda}(t)) m(t) \end{aligned}$$

Combining (5) and (1),

$$(6) \quad P(\psi') - P(\psi) \leq \sum_{t=0}^{\tau} (\bar{\lambda}(t) - \underline{\lambda}(t)) m(t) \equiv \Delta(\psi)$$

A formula similar to (5) shows that (6) must also hold in the continuous case.

Now  $\psi'$  is not needed to compute any of the quantities on the right-hand side of (6), so every effort density  $\psi$  has associated with it an upper bound on the detection probability  $P(\psi) + \Delta(\psi)$ . In the event  $\bar{\lambda}(t) = \underline{\lambda}(t)$  for all  $t$ ,  $\psi$  must actually be optimal--this has been observed by Stone [1], and in fact our whole development is a modification of his



sufficiency proof. The main issue is now computational: is determination of  $\Delta(\psi)$  worth the effort?

### 3. THE CASE OF MARKOV MOTION AND EXPONENTIAL DETECTION FUNCTION.

Let

$$Z = Z_t^- + w(X_t, t) \psi(X_t, t) + Z_t^+ \quad \text{for } 0 \leq t \leq \tau,$$

where

$$Z_t^- = \sum_{u=0}^{t-1} w(X_u, u) \psi(X_u, u),$$

$$Z_t^+ = \sum_{u=t+1}^{\tau} w(X_u, u) \psi(X_u, u),$$

and

$$Z_0^- = Z_\tau^+ = 0.$$

Then

$$D_\tau(\psi, x, t) = w(x, t) p_t(x) E(s(Z_t^- + w(x, t) \psi(x, t) + Z_t^+) | X_t = x).$$

If the motion is Markov, then  $Z_t^+$  is independent of  $Z_t^-$  when  $X_t = x$  is given. If  $b(z) = 1 - \exp(-z)$ , then  $s(z) = \exp(-z)$ . If both conditions hold, then

$$D_\tau(\psi, x, t) = w(x, t) P(\psi, x, t) \exp(-w(x, t) \psi(x, t)) Q(\psi, x, t),$$

where

$$P(\psi, x, t) \equiv p_t(x) E(s(Z_t^-) | X_t = x)$$

and

$$Q(\psi, x, t) \equiv E(s(Z_t^+) | X_t = x).$$

$P(\psi, x, t)$  is the joint probability that  $X_t = x$  and that the target is not detected by any of the searches at  $0, 1, \dots, t-1$ ; note that  $P(\psi, x, t)$  does not depend on  $\psi(y, u)$  for  $u \geq t$ .  $Q(\psi, x, t)$  is the conditional probability that the target is not detected by any of the searches at  $t+1, \dots, \tau$  given that  $X_t = x$ ; note that  $Q(\psi, x, t)$  does not depend on  $\psi(y, u)$  for  $u \leq t$ . Given  $\psi$ ,  $P(\psi, x, t)$  and  $Q(\psi, x, t)$  can be easily computed recursively.  $P(\psi, x, 0)$  is a given initial distribution, and  $P(\psi, x, t+1)$  can be obtained from  $P(\psi, x, t)$  using the Markov transition rule and  $\psi(\cdot, t)$ . Similarly,  $Q(\psi, x, \tau) \equiv 1$ , and  $Q(\psi, x, t-1)$  can be obtained from  $Q(\psi, x, t)$  using the Markov transition rule and  $\psi(\cdot, t)$ . After obtaining  $P(\psi, x, t)$  and  $Q(\psi, x, t)$ , it is a simple matter to compute  $D_\tau(\psi, x, t)$  and then  $\Delta(\psi)$ .

Algorithms for finding the optimal effort distribution  $\psi^*$  typically operate by generating a sequence  $\psi_1, \psi_2, \dots$  that approaches  $\psi^*$ . The method of computation is such that  $\Delta(\psi_1)$  can be computed with only slightly more effort. Consider the discrete case. For any  $t$ , the probability of detection is

$$(7) \quad P(\psi) = 1 - \int_x P(\psi, x, t) \exp(-w(x, t) \psi(x, t)) Q(\psi, x, t)$$

In order to find  $\psi^*$ , we first make an initial guess  $Q^0(x, t)$  and then calculate  $\psi^r$  the function that minimizes

$$\int_x P(\psi, x, t) \exp(-w(x, t) \psi(x, t)) Q^{n-1}(x, t) \quad \text{for } n \geq 1,$$

with  $Q^n(x, t) = Q(\psi^n, x, t)$  for  $n \geq 1$ . Each of the minimization problems is relatively simple, and it can be shown that  $P(\psi^n)$  increases with  $n$  [1, 2, 3].

For each of the minimization problems, there must exist a function  $\lambda_n(t)$  such that  $D_n(x, t) = \lambda_n(t)$  when  $\psi^n(x, t) > 0$  and  $D_n(x, t) \leq \lambda_n(t)$  when  $\psi^n(x, t) = 0$ , where

$$D_n(x, t) = w(x, t) P(\psi^n, x, t) \exp(-w(x, t) \psi^n(x, t)) Q^{n-1}(x, t).$$

These are simply the Kuhn-Tucker conditions; in practice, some sort of a search is made until  $\lambda_n(t)$  is found such that  $\int_x \psi(x, t) = m(t)$ . But

$$(8) \quad D_T(\psi^n, x, t) \begin{cases} = \lambda_n(t) Q^n(x, t)/Q^{n-1}(x, t) & \text{when } \psi^n(x, t) > 0 \\ \leq \lambda_n(t) Q^n(x, t)/Q^{n-1}(x, t) & \text{when } \psi^n(x, t) = 0 \end{cases}$$

so computation of  $D_T(\psi^n, x, t)$  is a small burden given that  $\lambda_n(t)$ ,  $Q^n(x, t)$ , and  $Q^{n-1}(x, t)$  have to be computed in any case.

In the continuous case,  $\int_x$  is replaced by  $\int dx$ ; otherwise, the development remains the same.

#### 4. AN EXAMPLE

We set  $\tau = 79$ , so there are 80 looks. The target does a random walk over the cells  $1, \dots, 67$ , starting at cell 34. At each opportunity, it moves left with probability .3, right with probability .3, or else does not move. The boundary is reflecting. We set  $w(x,t) = .001625$  and  $m(t) = 100$ ; if all 100 units of search effort are used in a single cell, the probability of detection would be  $1 - \exp(-.1625) = .15$  at each of the 80 opportunities. We also set  $Q^0(x,t) \equiv 1$ .

The first five detection probabilities (and values for  $\Delta(\psi)$ ) are: .7054 (.0393), .7075 (.0246), .7086 (.0120), .7090 (.0067), and .7092 (.0036). The amounts of CPU time required on the NPS IBM 360/67 for computation of each of these five pairs were 2.4, 2.8, 2.4, 2.0, and 2.0 seconds, respectively. The second pass always takes a relatively long time because  $Q^1(x,t)$  is radically different from  $Q^0(x,t)$ . The function  $\exp(-w(x,t) \psi^5(x,t))$  is shown to three decimal places in Figure 1, with \*\*\*\* indicating no search. Time reads down the page (there are 80 rows). Only the central cells are shown; the rest are not searched.

It is interesting to compare these results with those that result when all effort must be placed in a single cell at each time. The best probability of detection in that case is apparently .64--sufficient conditions for optimality are not known in that case, but the author has done sufficient

experimentation to conjecture that .64 is the answer. The increase from .64 to .7092 can, of course, be attributed to the relaxation of a constraint.

## REFERENCES

- (1) "Numerical Optimization of Search for a Moving Target," Daniel H. Wagner, Associates Report to Office of Naval Research, by L. D. Stone et al., 23 June 1978.
- (2) "Optimal Search for a Moving Target in Discrete Time and Space," by S. S. Brown (submitted to Opns. Res.).
- (3) "On Search for a Moving Target," by A. R. Washburn (submitted to NRLQ).

FIGURE 1. Survival Probabilities for Near Optimal Search.

980	981	982	983	984	985	986	987	988	989	990
991	992	993	994	995	996	997	998	999	1000	1001
1002	1003	1004	1005	1006	1007	1008	1009	1010	1011	1012
1013	1014	1015	1016	1017	1018	1019	1020	1021	1022	1023
1024	1025	1026	1027	1028	1029	1030	1031	1032	1033	1034
1035	1036	1037	1038	1039	1040	1041	1042	1043	1044	1045
1046	1047	1048	1049	1050	1051	1052	1053	1054	1055	1056
1057	1058	1059	1060	1061	1062	1063	1064	1065	1066	1067
1068	1069	1070	1071	1072	1073	1074	1075	1076	1077	1078
1079	1080	1081	1082	1083	1084	1085	1086	1087	1088	1089
1090	1091	1092	1093	1094	1095	1096	1097	1098	1099	1100
1101	1102	1103	1104	1105	1106	1107	1108	1109	1110	1111
1112	1113	1114	1115	1116	1117	1118	1119	1120	1121	1122
1123	1124	1125	1126	1127	1128	1129	1130	1131	1132	1133
1134	1135	1136	1137	1138	1139	1140	1141	1142	1143	1144
1145	1146	1147	1148	1149	1150	1151	1152	1153	1154	1155
1156	1157	1158	1159	1160	1161	1162	1163	1164	1165	1166
1167	1168	1169	1170	1171	1172	1173	1174	1175	1176	1177
1178	1179	1180	1181	1182	1183	1184	1185	1186	1187	1188
1189	1190	1191	1192	1193	1194	1195	1196	1197	1198	1199
1200	1201	1202	1203	1204	1205	1206	1207	1208	1209	1210
1211	1212	1213	1214	1215	1216	1217	1218	1219	1220	1221
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1233	1234	1235	1236	1237	1238	1239	1240	1241	1242	1243
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